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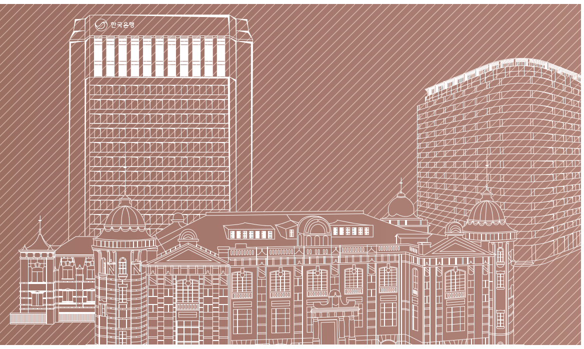
A Properly Ordered Zero Sign Restrictions on VARX for a Small Open Economy



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A Properly Ordered Zero Sign Restrictions on VARX for a Small Open Economy

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Contents

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Responses to shocks of endogenous variables and those to shocks of exogenous variables are mixed when rotation matrices are used for the sign restriction in VARX model. The mixing of responses to incompatible shocks may result in erroneous identifications when employing sign restrictions in the context of VARX frameworks.

The order in which Givens rotation matrices are multiplied introduces an additional degrees of freedom, a consequence of the non-abelian nature of matrix multiplication. This paper elucidates that the multiplicative order of Givens matrices significantly affects the identification of structural shocks, particularly in settings where sign restrictions are imposed within VARX models that incorporate exogenous variables.

To address these identification challenges, we propose a dual methodology: a systematic ordering of Givens rotation matrices combined with zero restrictions on these appropriately ordered matrices. Both approaches are essential for the accurate identification of structural shocks within VARX models. We designate the ordered Givens matrices as Properly Ordered Givens (POG) matrices and refer to the zero-restricted POG matrices as Zero-Restricted Properly Ordered Givens (ZPOG) matrices.

As a case study, this paper investigates the impact of inflation expectations on the pass-through effect of the Consumer Price Index (CPI) through a counterfactual analysis. The estimation results indicate that the inflation expectations channel amplifies the response of the CPI to shocks originating in the oil market.

Measures of inflation expectations serve as a key signal of the credibility of central bank policies. Consequently, it is imperative for monetary authorities to actively monitor and stabilize inflation expectations to mitigate the potential amplifying effects associated with inflation expectations, thereby enhancing the credibility of their monetary policy.

Keywords: Exogeneity, Sign Restrictions, Zero Restrictions

JEL Classification: C13, C21, C23

I. Introduction

Givens rotation matrices are commonly employed in SVAR models to identify structural shocks. It is well established that each individual Givens matrix has degrees of freedom, as the values of the elements depend on the rotation angles. Different rotation angles result in different element values for each individual Givens matrix.

However, the multiplication of multiple Givens matrices introduces a different kind of degrees of freedom: similar to ordinary matrix multiplication, the multiplication of Givens matrices does not satisfy the commutative law. Givens matrices are non-abelian, meaning the order of their multiplication can significantly impact the outcome, especially when analyzing VARX models that include exogenous variables.

In this paper, we demonstrate that the order of Givens matrix multiplication can be crucial when identifying structural shocks with sign restrictions in VARX models, which include exogenous variables. To properly identify structural shocks to both endogenous and exogenous variables in VARX model, it is essential to combine zero restrictions with a properly ordered multiplication of Givens rotation matrices. Givens matrices mix the coordinates of the variables in VAR or VARX models. Thus, when arbitrarily ordered multiplications of Givens matrices are used to identify structural shocks in VARX models, the responses of endogenous variables and exogenous variables are arbitrarily mixed. This implies that exogenous variables may respond to shocks of endogenous variables due to the mixing of the responses.

This presents a challenge: exogenous variables in VARX models must not respond to shocks affecting endogenous variables. For instance, international oil prices should not respond to shocks in domestic prices of a small open economy. However, arbitrarily ordered multiplications of Givens matrices result in a mixture of responses of exogenous and endogenous variables to shocks, leading to incorrect identification. Therefore, it is necessary to isolate the responses of exogenous variables from those of endogenous variables. Furthermore, it is critical to choose appropriate rotation angles to ensure that exogenous variables do not respond to shocks affecting endogenous variables.

Haberis and Sokol (2014) and Arias et al. (2018) consider the utilization of the combinations of sign restrictions and zero restrictions in the context of an identification strategy of VAR model. Haberis and Sokol (2014) use a combination of

Givens and Householder transformations, while Arias et al, (2018) use sole Householder transformation for the identification. However, these strategies may not be appropriate in the context of VARX models in which the responses of endogenous and exogenous variables are arbitrarily mixed.

We apply the properly ordered sign restrictions strategy to a VECM with Exogenous Variables (VECMX) to examine the effects of three types of oil market shocks on Korean CPI and the role of inflation expectations in the price transmission process.

The remainder of the paper is structured as follows. Section II discusses local projection estimation for the VECMX model containing I(0) time series. In Section III, we propose the properly ordered sign restrictions approach for VARX models. Section IV provides an illustrative application, and Section V concludes.

II. Local Projections for VECMX Model

1. VECM Model with Exogenous Variables

Let $\mathbf{y}_t^1$ be $(N_1 \times 1)$ vector of $I(1)$ endogenous domestic variable, $\mathbf{y}_t^0$ be $(N_0 \times 1)$ vector of $I(0)$ endogenous domestic variables, $\mathbf{x}_t$ be $(N_x \times 1)$ vector of $I(1)$ exogenous global variables, and $\mathbf{y}_t = \begin{pmatrix} \mathbf{y}_t^{1'} & \mathbf{y}_t^{0'} \end{pmatrix}'$. Consider a VARX model:

$$\mathbf{z}_t = \sum_{i=1}^{p} \Phi_i \mathbf{z}_{t-i} + e_t, \tag{1}$$

where

$$\mathbf{z}_t = \begin{pmatrix} \mathbf{y}_t^1 \\ \mathbf{y}_t^0 \\ \mathbf{x}_t \end{pmatrix}, \quad \Phi_i = \begin{pmatrix} \Phi_i^{11} & \Phi_i^{10} & \Phi_i^{1x} \\ \Phi_i^{01} & \Phi_i^{00} & \Phi_i^{0x} \\ \Phi_i^{x1} & \Phi_i^{x0} & \Phi_i^{xx} \end{pmatrix}. \tag{2}$$

We make the following assumption.

Assumption 1.

i) There are cointegrating relationships between $\mathbf{y}_t^1$ such as $\mathbf{v}_t = \beta_1' \mathbf{y}_t^1$, while $\mathbf{y}_t^0$ is exogenous to the cointegrating vectors.

ii) $\mathbf{x}_t$ is a vector of exogenous global variables that are not affected by endogenous domestic variables $\mathbf{y}_{t-i}$ for $i = 0, 1, \cdots$.

ii) of Assumption 1 implies that $\mathbf{x}_t$ is strongly exogenous to $\mathbf{y}_t$. We have a VECMX from Assumption 1[1]:

$$\mathbf{z}_t = \sum_{i=1}^{p-1} \Gamma_i \mathbf{w}_{t-i} + \Pi \mathbf{z}_{t-1} + \mathbf{e}_t, \tag{3}$$

where $\mathbf{w}_t = \begin{pmatrix} \Delta \mathbf{y}_t^{1'} & \mathbf{y}_t^{0'} & \Delta \mathbf{x}_t^{'} \end{pmatrix}^{'}$, $\mathbf{e}_t = \begin{pmatrix} \mathbf{e}_{1t}^{'}, & \mathbf{e}_{0t}^{'}, & \mathbf{e}_{xt}^{'} \end{pmatrix}^{'}$, $\Gamma_i = -[\Phi_{i+1} + \cdots + \Phi_p]$,

$$\Gamma_i = \begin{pmatrix} \Gamma_i^{11} & \Gamma_i^{10} & \Gamma_i^{1x} \\ \Gamma_i^{01} & \Gamma_i^{00} & \Gamma_i^{0x} \\ \Gamma_i^{x1} & \Gamma_i^{x0} & \Gamma_i^{xx} \end{pmatrix} \tag{4}$$

for $i = 1, \cdots, p-1$, $\Pi^{11} - \mathbf{I} = -\Phi^{11}(1) = -\left(\alpha_1 \beta_1^{'}\right)$ and β_1 and α_1 is $(N_1 \times r)$ vectors, respectively. According to the ii) of Assumption 1, $\Gamma_i^{x1} = \Gamma_i^{x0} = \mathbf{0}$ for $i = 1, \cdots, p-1$ since $\mathbf{x}$ is not affected by the endogenous domestic variables $\mathbf{y}_t$.

The usual assumption in VAR models is that the errors are distributed as

$$\mathbf{e}_t = \begin{pmatrix} \mathbf{e}_{yt} \\ \mathbf{e}_{xt} \end{pmatrix} \sim i.i.d. N\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{pmatrix} \right), \tag{5}$$

where $\mathbf{e}_{yt} = \begin{pmatrix} \mathbf{e}_{1t}^{'} & \mathbf{e}_{0t}^{'} \end{pmatrix}^{'}$. The strong exogeneity of $\mathbf{x}_t$ implies that the error vectors $\mathbf{e}_{yt}$ and $\mathbf{e}_{xt}$ are independent. Thus,

$$\mathbf{e}_t \sim i.i.d. N\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \Sigma_{yy} & \mathbf{0} \\ \mathbf{0} & \Sigma_{xx} \end{pmatrix} \right) \tag{6}$$

since $\Sigma_{yx} = \mathbf{0}$ and $\Sigma_{xy} = \Sigma_{yx}^{'} = \mathbf{0}$.

1) Refer Hamilton (1994), p. 652.

Subtract $\left(\mathbf{y}_{t-1}^{1'}, \quad \mathbf{0}', \quad \mathbf{x}_{t-1}'\right)'$ on both sides of (3), then under Assumption 1,

$$\begin{aligned}\mathbf{w}_t &= \sum_{i=1}^{p-1} \Gamma_i \mathbf{w}_{t-i} + (\Pi - \mathbf{I})\,\mathbf{z}_{t-1} + \mathbf{e}_t \\ &= -\alpha\beta_1'\mathbf{y}_{t-1}^1 + \sum_{i=1}^{p-1} \Gamma_i \mathbf{w}_{t-i} + \mathbf{e}_t,\end{aligned} \tag{7}$$

where $\alpha = \left(\alpha_1', \quad \alpha_0', \quad \mathbf{0}'\right)'$ is $((N_1 + N_0 + N_x) \times r)$ vector, α_1 is $(N_1 \times r)$ vector, and α_0 is $(N_0 \times r)$ vector. More precisely, we have

$$\begin{aligned}\Delta\mathbf{y}_t^1 &= -\alpha_1 \mathbf{v}_{t-1} + \sum_{i=1}^{p-1} \Gamma_i^{11} \Delta\mathbf{y}_{t-i}^1 + \sum_{i=1}^{p-1} \Gamma_i^{10} \mathbf{y}_{t-i}^0 + \sum_{i=1}^{p-1} \Gamma_i^{1x} \Delta\mathbf{x}_{t-i} + e_{1t}, \\ \mathbf{y}_t^0 &= -\alpha_0 \mathbf{v}_{t-1} + \sum_{i=1}^{p-1} \Gamma_i^{01} \Delta\mathbf{y}_{t-i}^1 + \sum_{i=1}^{p-1} \Gamma_i^{00} \mathbf{y}_{t-i}^0 + \sum_{i=1}^{p-1} \Gamma_i^{0x} \Delta\mathbf{x}_{t-i} + e_{0t}, \\ \Delta\mathbf{x}_t &= \sum_{i=1}^{p-1} \Gamma_i^{xx} \Delta\mathbf{x}_{t-i} + e_{xt}.\end{aligned} \tag{8}$$

2. Local Projections for VECMX Model

In order to analyse the dynamic responses of (8), we construct a state space representation.[2)] β_1' can be used in (8) due to $\mathbf{v}_t = \beta_1'\mathbf{y}_t^1$. By premultiplying β_1' to the first equation,

$$\Delta\mathbf{v}_t = -\beta_1'\alpha_1\mathbf{v}_{t-1} + \beta_1' \sum_{i=1}^{p-1} \Gamma_i^{11} \Delta\mathbf{y}_{t-i}^1 + \beta_1' \sum_{i=1}^{p-1} \Gamma_i^{10} \mathbf{y}_{t-i}^0 + \beta_1' \sum_{i=1}^{p-1} \Gamma_i^{1x} \Delta\mathbf{x}_{t-i} + \beta_1' e_{1t}. \tag{9}$$

Thus,

$$\mathbf{v}_t = \left(I - \beta_1'\alpha_1\right)\mathbf{v}_{t-1} + \beta_1' \sum_{i=1}^{p-1} \Gamma_i^{11} \Delta\mathbf{y}_{t-i}^1 + \beta_1' \sum_{i=1}^{p-1} \Gamma_i^{10} \mathbf{y}_{t-i}^0 + \beta_1' \sum_{i=1}^{p-1} \Gamma_i^{1x} \Delta\mathbf{x}_{t-i} + \beta_1' e_{1t}. \tag{10}$$

2) Chong et al. (2012) use local projections for estimation of VECM.

The state space representation of (8) and (10) is

$$\begin{pmatrix} \mathbf{v}_t \\ \mathbf{w}_t \\ \mathbf{w}_{t-1} \\ \vdots \\ \mathbf{w}_{t-p} \end{pmatrix} = \begin{pmatrix} \left(\mathbf{I}-\beta_1'\alpha_1\right) & \beta'\Gamma_1 & \cdots & \beta'\Gamma_{p-2} & \beta'\Gamma_{p-1} \\ -\alpha & \Gamma_1 & \cdots & \Gamma_{p-2} & \Gamma_{p-1} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v}_{t-1} \\ \mathbf{w}_{t-1} \\ \mathbf{w}_{t-2} \\ \vdots \\ \mathbf{w}_{t-p+1} \end{pmatrix} + \begin{pmatrix} \beta_1' e_{1t} \\ \mathbf{e}_t \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}, \tag{11}$$

where $\beta' = \left(\beta_1', \quad \mathbf{0}', \quad \mathbf{0}'\right)'$. A more compact form of (11) is

$$\zeta_t = \Gamma^* \zeta_{t-1} + \eta_t, \tag{12}$$

where

$$\zeta_t = \begin{pmatrix} \mathbf{v}_t \\ \mathbf{w}_t \\ \mathbf{w}_{t-1} \\ \vdots \\ \mathbf{w}_{t-p} \end{pmatrix}, \Gamma^* = \begin{pmatrix} \left(\mathbf{I}-\beta_1'\alpha_1\right) & \beta'\Gamma_1 & \cdots & \beta'\Gamma_{p-2} & \beta'\Gamma_{p-1} \\ -\alpha & \Gamma_1 & \cdots & \Gamma_{p-2} & \Gamma_{p-1} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix}, \eta_t = \begin{pmatrix} \beta_1' e_{1t} \\ \mathbf{e}_t \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}.$$

Equation (12) allows to construct linear forecasts of $\mathbf{v}_{t+h}$ and $\Delta\mathbf{y}_{t+h}$ for $h = 1, \cdots, H$ given information set up to t. We have

$$\begin{aligned} \mathbf{v}_{t+h} &= \Gamma^{h*}_{[1,1]}\mathbf{v}_t + \Gamma^{h*}_{[1,2]}\Delta\mathbf{y}_t + \sum_{i=1}^{p}\Gamma^{h*}_{[1,i+2]}\Delta\mathbf{y}_{t-i} + \nu_{t+h}, \\ \mathbf{w}_{t+h} &= \Gamma^{h*}_{[2,1]}\mathbf{v}_t + \Gamma^{h*}_{[2,2]}\mathbf{w}_t + \sum_{i=1}^{p}\Gamma^{h*}_{[2,i+2]}\mathbf{w}_{t-i} + \mathbf{e}_{t+h}, \end{aligned} \tag{13}$$

for $h = 1, \cdots, H$, and $\nu_{t+h} = \beta_1' e_{1,t+h}$, $\Gamma^{h*}_{[i,j]}$ denotes the (i,j) block of the matrix Γ^* raised to the hth power. Thus, the impulse responses associated with the system in (8) due to a shock to the long-run equilibrium are

$$\begin{aligned} IRF\left(\mathbf{v}_{t+h}|\beta_1' e_{1,t+1} = I\right) &= E\left[\mathbf{v}_{t+h}|\beta_1' e_{1,t+1} = I\right] - E\left[\mathbf{v}_{t+h}|\beta_1' e_{1,t+1} = 0\right] \\ &= \Gamma^{h*}_{[1,1]} + \Gamma^{h*}_{[1,2]}\beta_1, \end{aligned} \tag{14}$$

$$IRF\left(\Delta\mathbf{y}_{t+h}|\beta_1'e_{1,t+1}=I\right)=E\left[\Delta\mathbf{y}_{t+h}|\beta_1'e_{1,t+1}=I\right]-E\left[\Delta\mathbf{y}_{t+h}|\beta_1'e_{1,t+1}=0\right]$$
$$=\Gamma^{h*}_{[2,1]}+\Gamma^{h*}_{[2,2]}\beta_1. \quad (15)$$

The impulse responses of $\mathbf{w}_t$ due to a shock to $\mathbf{w}_t$ is

$$IRF\left(\mathbf{w}_{t+h}|\mathbf{w}_t\right)=\begin{pmatrix}\mathbf{I}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}+\begin{pmatrix}\alpha_1\beta_1'&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}+\Gamma^{h*}_{[2,2]}. \quad (16)$$

We use local projection to estimate (8).

For a non-oil-producing small open economy, it is generally assumed that shocks to domestic endogenous variables, such as inflation or GDP, do not cause global exogenous variables, such as crude oil production or crude oil prices, to respond. In other words, exogenous global variables are unaffected by the domestic shocks of a small open economy. Suppose the dynamics of the small open economy can be represented by equation (8). The VECMX in equation (8) can be transformed into a VARX, as shown in equation (17):

$$\begin{aligned}\mathbf{y}_t^1&=\sum_{i=1}^{p}\Phi_i^{11}\mathbf{y}_{t-i}^1+\sum_{i=1}^{p}\Phi_i^{10}\mathbf{y}_{t-i}^0+\sum_{i=1}^{p}\Phi_i^{1x}\mathbf{x}_{t-i}+e_{1t},\\ \mathbf{y}_t^0&=\sum_{i=1}^{p}\Phi_i^{01}\mathbf{y}_{t-i}^1+\sum_{i=1}^{p}\Phi_i^{00}\mathbf{y}_{t-i}^0+\sum_{i=1}^{p}\Phi_i^{0x}\mathbf{x}_{t-i}+e_{0t},\\ \mathbf{x}_t&=\sum_{i=1}^{p}\Phi_i^{xx}\mathbf{x}_{t-i}+e_{xt},\end{aligned} \quad (17)$$

where $\Phi_1^{11}=\left(I-\alpha_1\beta_1'+\Gamma_1^{11}\right)$, $\Phi_p^{11}=-\Gamma_{p-1}^{11}$, $\Phi_i^{11}=\left(\Gamma_i^{11}-\Gamma_{i-1}^{11}\right)$ for $i=2,\cdots,p-1$, $\Phi_p^{10}=\mathbf{0}$, $\Phi_i^{10}=\Gamma_i^{10}$ for $i=1,\cdots,p-1$, $\Phi_1^{1x}=\Gamma_1^{1x}$, $\Gamma_p^{1x}=-\Gamma_{p-1}^{1x}$, $\Gamma_i^{1x}=\left(\Gamma_i^{1x}-\Gamma_{i-1}^{1x}\right)$ for $i=2,\cdots,p-1$, $\Phi_1^{01}=\left(-\alpha_1\beta_1'+\Gamma_1^{01}\right)$, $\Phi_p^{01}=-\Gamma_{p-1}^{01}$, $\Phi_i^{01}=\left(\Gamma_i^{01}-\Gamma_{i-1}^{01}\right)$ for $i=2,\cdots,p-1$, $\Phi_p^{00}=\mathbf{0}$, $\Phi_i^{00}=\Gamma_i^{00}$ for $i=1,\cdots,p-1$, $\Phi_1^{0x}=\Gamma_1^{0x}$, $\Phi_p^{0x}=-\Gamma_{p-1}^{0x}$, $\Phi_i^{0x}=\left(\Gamma_i^{0x}-\Gamma_{i-1}^{0x}\right)$ for $i=2,\cdots,p-1$, $\Phi_i^{x1}=\Phi_i^{x0}=\mathbf{0}$ for $i=1,\cdots,p$, $\Phi_1^{xx}=\Gamma_1^{xx}$, $\Phi_p^{xx}=-\Gamma_{p-1}^{xx}$, $\Phi_i^{xx}=\left(\Gamma_i^{xx}-\Gamma_{i-1}^{xx}\right)$ for $i=2,\cdots,p-1$. Thus, the matrix Φ_i in (2) have triangular matrix forms for $i = 1, \cdots, p$.

$$\Phi_i = \begin{pmatrix} \Phi_i^{11} & \Phi_i^{10} & \Phi_i^{1x} \\ \Phi_i^{01} & \Phi_i^{00} & \Phi_i^{0x} \\ \mathbf{0} & \mathbf{0} & \Phi_i^{xx} \end{pmatrix}. \tag{18}$$

3. Impulse Responses of VARX

Suppose that we have a VMA representation of (1):

$$\mathbf{z}_t = \sum_{i=1}^{\infty} \varphi_i \mathbf{e}_{t-i}, \tag{19}$$

where

$$\varphi_0 = I, \quad \varphi_i = \sum_{j=1}^{i} \varphi_{i-j}\Phi_j,$$
$$\varphi_i = \begin{pmatrix} \varphi_i^{11} & \varphi_i^{10} & \varphi_i^{1x} \\ \varphi_i^{01} & \varphi_i^{00} & \varphi_i^{0x} \\ \varphi_i^{x1} & \varphi_i^{x0} & \varphi_i^{xx} \end{pmatrix} \tag{20}$$

for $i = 1, 2, \cdots$. When the coefficient matrices $\Phi_i (i = 1, \cdots, p)$ in (1) are triangular matrices, the matrices of VMA in (20), φ_i also are triangular matrices for $i = 1, 2, \cdots$.

$$\varphi_i = \begin{pmatrix} \varphi_i^{11} & \varphi_i^{10} & \varphi_i^{1x} \\ \varphi_i^{01} & \varphi_i^{00} & \varphi_i^{0x} \\ \mathbf{0} & \mathbf{0} & \varphi_i^{xx} \end{pmatrix} \tag{21}$$

for $i = 1, 2, \cdots$. φ_i implies that exgenous global variables $\mathbf{x}$ do not respond to shocks of endogenous domestic variables $\mathbf{y}$ since $\varphi_i^{x1} = \varphi_i^{x0} = \mathbf{0}$ for $i = 1, 2, \cdots$.

III. Properly Ordered Zero Sign Restrictions

1. A Degrees of Freedom of Givens Matrix: Non-Abelianness of Multiplication of Matrices

Orthogonal matrices are commonly used for sign restrictions. These matrices can be generated through either the Givens transformation or the Householder transformation. In this paper, we focus exclusively on the Givens transformation.

In the Givens matrix, (i,i) element is $\cos\theta$, (i,j) element is $-\sin\theta$, (j,i) element is $\sin\theta$, and (j,j) element is $\cos\theta$. All other elements are either 0 or 1. For instance, in the case of a 4-dimensional matrix, the Givens matrix has the following form:

$$Q_{2,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

where $\theta \in (0,\ \pi/2)$ and $Q_{2,3}^{'}Q_{2,3} = Q_{2,3}Q_{2,3}^{'} = I_4$. Matrix $Q_{i,j}$ serves to rotate two coordinate axes, specifically axes i and j, thereby mixing the corresponding variables. The number of ways to select two variables from the total of N variables is given by $(N(N-1))$. Givens matrices are constructed such that each variable is paired exactly once. Consequently, if we select angles $\theta_{i,j}$ from the set $\{\theta_{i,j,1}, \cdots \theta_{i,j,M}\}$, the total number of drawn matrices of Q is $M^{(N(N-1))}$. The final Givens matrix Q is obtained by multiplying all individual matrices $Q_{i,j}$:

$$Q = \prod_{i=1}^{N-1} \prod_{j=i+1}^{N} Q_{i,j}(\theta_{i,j}), \tag{23}$$

where the product runs over all unique pairs of indices (i,j).

The values of the elements in the matrix Q fundamentally depend on the rotation angles, $\theta_{i,j}$. However, even when the angles $\theta_{i,j}$ are fixed at specific values, the elements of Q may exhibit different values depending on the order in which the matrix $Q_{i,j}$ are multiplied. This phenomenon arises from the non-abelian nature of rotation matrices. For example,

$$Q_{1,2}Q_{2,3} \neq Q_{2,3}Q_{1,2}. \tag{24}$$

Thus, the specific ordering of multiplication can significantly affect the final resulting matrix Q. Therefore, even if $\theta_{i,j}$ is fixed for each pair i and j, the number of ways to construct Q increases with the number of arrangements of the $Q_{i,j}$ matrices. The number of ways to rearrange $Q_{i,j}$ to create Q is given by $[(N(N-1))/2]!$. Therefore, in practice, the number of ways to create Q matrix is $M^{(N(N-1))/2} \times [(N(N-1))/2]!$.

As is well known, the matrix Q can also be constructed using the Householder transformation or the QR algorithm. Generally, there are N^2 elements in Q, but among these, $N(N+1)/2$ elements are restricted. Consequently, there are total $N(N-1)/2$ free parameters, which correspond to the number of free parameters in the Givens matrix. This implies that the Givens matrix Q can be transformed into the Q matrix generated by the QR algorithm, and conversely, such a transformation is also feasible.

As the number of variables, N, increases, the computational cost of constructing a Givens matrix grows exponentially. In contrast, the QR transformation does not exhibit the same exponential growth in cost, giving the QR method a relative advantage. For this respect, the QR method is often preferred over the Givens method in practice. However, it is worth noting that the Householder QR method may not be appropriate for the identification of VARX that incorporates exogenous variables.

2. Impulse Responses of VARX Under Properly Ordered Givens Matrices

To implement sign restrictions, it is essential to have uncorrelated structural shocks with unit variances. The structural model corresponding to equation (1) can be expressed as follows:

$$B_0\mathbf{z}_t = \sum_{i=1}^{p} B_i\mathbf{z}_{t-i} + \nu_t, \tag{25}$$

where $e_t = B_0^{-1}v_t$, $B_i = B_0^{-1}\Phi_i$ for $i = 1, \cdots, p-1$. The structure matrix B_0 for a small open economy that has strong exogenous variables must take the form of an upper triangular matrix.

Consider a structure:

$$\hat{\Sigma} = PP', \tag{26}$$

$$P = \begin{bmatrix} P_y & \mathbf{0} \\ \mathbf{0} & P_x \end{bmatrix}, \quad \hat{\Sigma}_{yy} = P_y P_y', \quad \hat{\Sigma}_{xx} = P_x P_x'. \tag{27}$$

Define a structural shock, $v_t = P^{-1}\mathbf{e}_t$, then v_t is uncorrelated and has unit variance. Note that P is upper triangular matrix even though P_y and P_x are lower triangular matrices.[3)] Thus, $\varphi_i P$ is upper triangular matrix. This implies that the exogenous global variables $\mathbf{x}_t$ do not respond to the shocks of the endogenous domestic variables $\mathbf{y}_t$:

$$IRF_i^{SVARX} = \Psi_i v_t = \begin{bmatrix} \Psi_i^{yy} & \Psi_i^{yx} \\ 0 & \Psi_i^{xx} \end{bmatrix} \begin{bmatrix} v_t^{endo} \\ v_t^{exo} \end{bmatrix}, \tag{28}$$

where

$$\Psi_i = \varphi_i P = \begin{bmatrix} \varphi_i^{yy} & \varphi_i^{yx} \\ 0 & \varphi_i^{xx} \end{bmatrix} \begin{bmatrix} P_y & \mathbf{0} \\ \mathbf{0} & P_x \end{bmatrix} = \begin{bmatrix} \varphi_i^{yy} P_y & \varphi_i^{yx} P_x \\ 0 & \varphi_i^{xx} P_x \end{bmatrix} = \begin{bmatrix} \psi_i^{yy} & \psi_i^{yx} \\ 0 & \psi_i^{xx} \end{bmatrix},$$

$v_t = \begin{bmatrix} v_t^{endo} \\ v_t^{exo} \end{bmatrix}$, v_t^{endo} represents structural shocks of the endogenous variables, and v_t^{exo} represents structural shocks of the exogenous variables. Note that v_t^{endo} and v_t^{exo} are independent.

Consider an arbitrary Givens matrix Q constructed in a manner analogous to (23) which is often used for sign restrictions. Let N represent the endogenous domestic

3) P is also lower triangular matrix since $\Sigma_{xy} = 0$ and $\Sigma_{yx} = 0$.

block and X the exogenous global block, respectively. Then

$$Q = \begin{bmatrix} Q_N & Q_{NX} \\ Q_{XN} & Q_X \end{bmatrix}, \tag{29}$$

where Q_N and Q_X are square matrices and have inverse matrix, respectively. If $Q^{'}Q = I$, we have the following three equations.

$$Q_N^{'}Q_N + Q_{XN}^{'}Q_{XN} = I,\ Q_N^{'}Q_{NX} + Q_{XN}^{'}Q_X = 0,\ Q_{NX}^{'}Q_{NX} + Q_X^{'}Q_X = I. \tag{30}$$

Suppose that impulse response matrices, Ψ_i are upper triangular matrices for $i = 1, 2, \cdots$ and consider a Givens matrix Q for which $Q_{XN} \neq 0$ is applied to impose sign restrictions in VARX. Under these conditions, [**Proposition 1**] holds:

Proposition 1.

Exogenous variables **respond** to shocks of endogenous variables under the sign restrictions with Q in (29).

Proof.

$$\begin{aligned} IRF_i^{SIGN} &= \begin{bmatrix} \Psi_i^{yy} & \Psi_i^{yx} \\ 0 & \Psi_i^{xx} \end{bmatrix} QQ^{'} \begin{bmatrix} v_t^{endo} \\ v_t^{exo} \end{bmatrix} \\ &= \begin{bmatrix} \varphi_i^{yy} P_y Q_N + \varphi_i^{yx} P_x Q_{NX} & \varphi_i^{yy} P_y Q_{XN} + \varphi_i^{yx} P_x Q_X \\ \varphi_i^{xx} P_x Q_{NX} & \varphi_i^{xx} P_x Q_X \end{bmatrix} \begin{bmatrix} \tau_t^{endo} \\ \tau_t^{exo} \end{bmatrix}, \end{aligned} \tag{31}$$

where $\tau_t = Qv_t$.

The left lower block of the impulse response matrix of VARX under sign restrictions with Q is not 0 unless $\varphi_i^{xx} P_x Q_{NX} = 0$. □

[**Proposition 1**] implies that when conventional sign restrictions using a typical Q matrix constructed by (23) are applied for identification, the impulse response functions fail to satisfy a critical requisite condition for small open economies: Requisite that exogenous (global) variables **do not** respond to shocks affecting endogenous (domestic) variables. Consequently, the sign restrictions that employ standard

Givens matrices, often constructed by the way in (23), are inadequate for capturing important features inherent to small open economies.

The condition necessary for the impulse response matrices of $VARX$ to be upper triangular under sign restrictions is given by $\Psi_i^{xx} Q_{NX} = 0$. For this condition to hold, Q_{NX} must be 0. This condition is necessary for the impulse response matrices of $VARX$ to be an upper triangular matrix, given that $\Psi_i^{xx} \neq 0$. If $Q_{NX} = 0$, it follows that $Q_{XN} = Q_{NX}^{'} = 0$. If $Q_{NX} = 0$, the impulse response matrices of $VARX$ with sign restrictions are

$$IRF_i^{SIGN} = \begin{bmatrix} \Psi_i^{yy} Q_N & \Psi_i^{yx} Q_X \\ 0 & \Psi_i^{xx} Q_X \end{bmatrix} \begin{bmatrix} \tau_t^{endo} \\ \tau_t^{exo} \end{bmatrix}. \tag{32}$$

The impulse response matrix in (32) satisfies the condition that the responses of exogenous (global) variables $\mathbf{x}_t$ **do not respond** to shocks of endogenous (domestic) variables $\mathbf{y}_t$ under sign restrictions, contingent upon the condition that $Q_{NX} = 0$.

3. Sign Restrictions with Properly Ordered Givens Matrix

The rotation matrix serves to mix the coordinates of two variables by rotating them to the same size and in the same direction. In this context, the Givens matrix is commonly referred to as a rotation matrix. Depending on the order of multiplications in which the $Q_{i,j}$ matrices are multiplied, the elements of the resulting matrix Q yield different values. Therefore, the order of multiplication of Givens matrices introduces a distinct type of degrees of freedom, reflecting their non-abelian nature. Let the number of endogenous variables be N_n, the number of exogenous variables be N_x, and $N = N_n + N_x$. Suppose that a Givens matrix Q_{POG} is constructed through the ordered multiplication of Q_N, Q_X and Q_{NX}:

$$\begin{aligned} Q_{POG} \equiv Q_N Q_X Q_{NX} &= \begin{bmatrix} Q_N^0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Q_X^0 \end{bmatrix} \begin{bmatrix} Q_{NX}^{11} & Q_{NX}^{12} \\ Q_{NX}^{21} & Q_{NX}^{22} \end{bmatrix} \\ &= \begin{bmatrix} Q_N^0 Q_{NX}^{11} & Q_N^0 Q_{NX}^{12} \\ Q_X^0 Q_{NX}^{21} & Q_X^0 Q_{NX}^{22} \end{bmatrix}, \end{aligned} \tag{33}$$

where

$$Q_N = \begin{bmatrix} Q_N^0 & 0 \\ 0 & I \end{bmatrix}, \; Q_X = \begin{bmatrix} I & 0 \\ 0 & Q_X^0 \end{bmatrix}, \; Q_{NX} = \begin{bmatrix} Q_{NX}^{11} & Q_{NX}^{12} \\ Q_{NX}^{21} & Q_{NX}^{22} \end{bmatrix}. \tag{34}$$

Q_N^0 is a $N_n \times N_n$ rotation matrix, which is formed by multiplying all rotation matrices that exclusively rotate the pairs of endogenous N_n variables. Similarly, let Q_X^0 be a $N_x \times N_x$ rotation matrix, generated by multiplying all rotation matrices that exclusively rotate the pairs of exogenous N_x variables. The matrix Q_{NX} is a rotation matrix that operates on two variables, consisting of one endogenous variable and one exogenous variable. The term POG signifies that the Givens matrix Q_{POG} is constructed through the properly ordered multiplications of Givens matrices. Consequently, the elements of Q_{POG} differ from those of Q which is constructed by the way of (23).

By using Q_{POG} as a Givens matrix, we obtain the impulse response matrix:

$$\begin{aligned} & IRF_i^{POG} \\ &= \begin{bmatrix} \Psi_i^{yy} & \Psi_i^{yx} \\ 0 & \Psi_i^{xx} \end{bmatrix} Q_{POG} \begin{bmatrix} \tau_t^{endo} \\ \tau_t^{exo} \end{bmatrix} \\ &= \begin{bmatrix} \varphi_i^{yy} P_y Q_N^0 Q_{NX}^{11} + \varphi_i^{yx} P_x Q_X^0 Q_{NX}^{21} & \varphi_i^{yy} P_y Q_N^0 Q_{NX}^{12} + \varphi_i^{yx} P_x Q_X^0 Q_{NX}^{22} \\ \varphi_i^{xx} P_x Q_X^0 Q_{NX}^{21} & \varphi_i^{xx} P_x Q_X^0 Q_{NX}^{22} \end{bmatrix} \begin{bmatrix} \tau_t^{endo} \\ \tau_t^{exo} \end{bmatrix}, \end{aligned} \tag{35}$$

Consider the interpretation of blocks in (35). The term, $\Psi_i^{yy} Q_N^0 Q_{NX}^{11}$, in the left upper corner represents a combination of the responses of endogenous variables to shocks originating from endogenous variables shocks, as captured by Ψ_i^{yy}. Similarly, $\Psi_I^{yx} Q_X^0 Q_{NX}^{21}$ in the left upper corner reflects the responses of endogenous variables to shocks originating from exogenous variables, represented by Ψ_i^{yx}. Consequently, the left upper block encapsulates the sum of these two responses of endogenous variables.

The right upper block also comprises the sum of responses of endogenous variables to shocks from both endogenous and exogenous variables. In contrast, the right lower block, $\Psi_I^{xx} Q_X^0 Q_{NX}^{22}$ signifies the responses of exogenous variables to shocks that solely originated from exogenous variables.

Under the setting of (1), the left lower block represents responses of exogenous variables to shocks of endogenous variables. However, the matrix, $\Psi_i^{xx}Q_X^0Q_{NX}^{21} = \varphi_i^{xx}P_xQ_X^0Q_{NX}^{21}$, in the left lower block of (35), illustrates a mixture of elements representing the responses of exogenous variables to shocks from exogenous variables, captured by Ψ_i^{xx}. Thus, this block does not include any information regarding the shocks to endogenous variables, thereby, lacking meaningful economic interpretations. The lack of information in the left lower block of (35) is a prerequisite for a proper impulse responses analysis for small open economies since we can just ignore the block without any serious interpretational considerations on the block.

4. Properly Ordered Zero Sign Restrictions

Equation (35) implies that the impulse response matrix IRF_i^{POG} is not upper triangular unless $P_xQ_X^0Q_{NX}^{21} = 0$. While the sign restriction imposed by Q_{POG} offers proper economic interpretations, it is essential to find the matrix such that $P_xQ_X^0Q_{NX}^{21} = 0$ to ensure that the left lower block of IRF_i^{POG} equals 0. Note that QL decomposition, in general, does not guarantee a 0 left lower block of Q matrix. Consequently, generating a rotation matrix using the Householder method often makes it challenging to restrict the left lower block to be a zero matrix which is necessary for proper sign restrictions on VARX.

The Givens method has degrees of freedom: rotation angles. Thus, we leverage the degrees of freedom, along with the different degrees of freedom introduced by the non-abelian nature of matrix multiplications, to find a rotation angle that results in the left lower block being a zero matrix. In general, to impose a zero restriction on an element $(i,\ j)$ of Q_{POG}, we first select a rotation matrix $Q_{POG}(\theta)$ and formulate an equation involving the rotation angle θ. We then solve for θ. This approach can be applied similarly when there are multiple restricted elements in Q_{POG}. Thus, the solution to the equation can be found easily. We derive the solution from the matrix Q_{POG} with a Cholesky decomposed covariance matrix, P. Let us denote the solution matrix Q_{POGC}.

Let θ_{ij} be the rotation angle that rotates the variables i and j, then

$$Q_{POGC}\left(\theta^{R}\right) = P\left(\prod_{\theta^{R}_{(i,j)}\in\Theta^{R}} Q_{\theta_{(i,j)}}\left(\theta^{R}\right)\right)\left(\prod_{\theta_{(i,j)}\notin\Theta^{R}} Q_{\theta_{(i,j)}}\right), \tag{36}$$

where Θ^{R} is a set of $\theta^{R}_{(i,j)}$. The product of $Q_{i,j}$ and $Q_{\ell,m}$ is

$$Q_{i,j}Q_{\ell,m} = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cos\left(\theta_{i,j}\right) & 0 & -\sin\left(\theta_{i,j}\right) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \sin\left(\theta_{i,j}\right) & 0 & \cos\left(\theta_{i,j}\right) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cos\left(\theta_{\ell,m}\right) & 0 & -\sin\left(\theta_{\ell,m}\right) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \sin\left(\theta_{\ell,m}\right) & 0 & \cos\left(\theta_{\ell,m}\right) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

The elements of the product $\left(Q_{i,j}Q_{\ell,m}\right)$ consist of entries that are either 0, 1 or products of $\cos\left(\theta_{i,j}\right), \sin\left(\theta_{i,j}\right), \cos\left(\theta_{\ell,m}\right)$ and $\sin\left(\theta_{\ell,m}\right)$. Consequently, the total number of unknowns equals to the number of matrices being multiplied, which is a direct function of the rotation angles, $\theta_{i,j}$ and $\theta_{\ell,m}$. This formulation guarantees that there always exists a solution to the equation constructed in this manner.

In general, we can assert that [**Proposition 2**] holds when we find a solution, θ^{R*}, which satisfies the condition:

$$P_{x}Q^{0}_{X}Q^{21}_{NX}\left(\theta^{R*}\right) = 0.$$

Proposition 2.

Exogenous (global) variables **x do not respond** to shocks of endogenous (domestic) variables under the sign restrictions with Q in (36).

Proof. In general, we can find a solution θ^{R*} which satisfies the condition $P_{x}Q^{0}_{X}Q^{21}_{NX}\left(\theta^{R*}\right) =$

0 that implies $\Psi_i^{xx}\left(Q_X^0 Q_{NX}^{21}\left(\theta^{R*}\right)\right)=0$ in (35). Thus,

$$Q_{ZPOGC}\left(\theta^{R*}\right)=\begin{bmatrix} P_y Q_N^0 Q_{NX}^{11}\left(\theta^{R*}\right) & P_y Q_X^0 Q_{NX}^{12}\left(\theta^{R*}\right) \\ 0 & P_x Q_X^0 Q_{NX}^{22}\left(\theta^{R*}\right) \end{bmatrix} \tag{37}$$

and

$$Q_{ZPOG}\left(\theta^{R*}\right)=\begin{bmatrix} Q_N^0 Q_{NX}^{11}\left(\theta^{R*}\right) & Q_X^0 Q_{NX}^{12}\left(\theta^{R*}\right) \\ 0 & Q_X^0 Q_{NX}^{22}\left(\theta^{R*}\right) \end{bmatrix} \tag{38}$$

under θ^{R*}. Therefore,

$$\begin{aligned} & IRF_i^{ZPOG}\left(\theta^{R*}\right) \\ &= \begin{bmatrix} \Psi_i^{yy} Q_N^0 Q_{NX}^{11}\left(\theta^{R*}\right)+\Psi_i^{yx} Q_X^0 Q_{NX}^{21}\left(\theta^{R*}\right) & \Psi_i^{yy} Q_N^0 Q_{NX}^{12}\left(\theta^{R*}\right)+\Psi_i^{yx} Q_X^0 Q_{NX}^{22}\left(\theta^{R*}\right) \\ 0 & \Psi_i^{xx} Q_X^0 Q_{NX}^{22}\left(\theta^{R*}\right) \end{bmatrix} \begin{bmatrix} \tau_t^{endo} \\ \tau_t^{exo} \end{bmatrix}. \end{aligned} \tag{39}$$

□

We define the Givens matrix, $Q_{ZPOG}\left(\theta^{R*}\right)$ as the zero restricted properly ordered Givens (ZPOG) matrices. Note that there exist trivial solutions θ^{R_0} that satisfy $Q_{NX}^{21}\left(\theta^{R_0}\right)=0$. Specifically, if either $\sin\left(\theta^{R_0}\right)=0$ or $\cos\left(\theta^{R_0}\right)=0$, then it trivially follows that:

$$Q_{NX}^{21}\left(\theta^{R_0}\right)=0.$$

The trivial solution leads to an undesirable outcome in which some columns of $Q_{NX}^{12}\left(\theta^{R_0}\right)$ and also $Q_X^0 Q_{NX}^{12}\left(\theta^{R_0}\right)$ become 0. As a result, both the left upper block and right upper block of (39) become distorted. When $Q_{NX}^{12}\left(\theta^{R_0}\right)=0$ hold, only the term $\Psi_i^{yy} Q_N^0 Q_{NX}^{11}\left(\theta^{R*}\right)$ remains in the left upper block since $\Psi_i^{yx} Q_X^0 Q_{NX}^{21}\left(\theta^{R*}\right)=0$. Consequently, the left upper block encapsulates a combination of solely endogenous responses. When $Q_{NX}^{12}\left(\theta^{R_0}\right)=0$ hold, the right upper block reflects a combination of exogenous responses and thus, distorted combination of responses of endogenous variables due to the distortion in $Q_N^0 Q_{NX}^{12}\left(\theta^{R_0}\right)$. Furthermore, some columns of impulse response matrix $IRF_i^{ZPOG}\left(\theta^{R*}\right)$ become 0 at all horizon. Consequently, in the process of selecting candidate Givens matrices, it is imperative to exclude these trivial solutions that result in $Q_{ZPOG}\left(\theta^{R_0}\right)$. By concentrating on non-trivial solutions, we ensure that the Givens matrices appropriately capture the intended structural relationships and retain meaningful economic interpretations.

Consider an example with two endogenous domestic variables and two strongly exogenous global variables. In this example, we need to impose zero restrictions on four specific elements of the Q_{POGC} matrix: $(3,1),(3,2),(4,1)$ and $(4,2)$. Consequently, we need to find four rotation angles: $\theta_{3,1},\theta_{3,2},\theta_{4,1}$ and $\theta_{4,2}$ to ensure that these elements of Q_{POGC} satisfy the required zero restrictions. In this example,

$$
\begin{aligned}
Q_{POGC}\left(\theta^{R}\right) &= P\, q\, Q_{NX}\left(\theta^{R}\right) \\
&= \begin{pmatrix} p_{y,11} & p_{y,12} & 0 & 0 \\ p_{y,21} & p_{y,22} & 0 & 0 \\ 0 & 0 & p_{x,33} & p_{x,34} \\ 0 & 0 & p_{x,43} & p_{x,44} \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 \\ 0 & 0 & q_{33} & q_{34} \\ 0 & 0 & q_{43} & q_{44} \end{pmatrix} \\
&\quad \begin{pmatrix} \cos\left(\theta_{1,4}\right) & 0 & 0 & -\sin\left(\theta_{1,4}\right) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\left(\theta_{1,4}\right) & 0 & 0 & \cos\left(\theta_{1,4}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\theta_{1,3}\right) & 0 & -\sin\left(\theta_{1,3}\right) & 0 \\ 0 & 1 & 0 & 0 \\ \sin\left(\theta_{1,3}\right) & 0 & \cos\left(\theta_{1,3}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\theta_{2,3}\right) & -\sin\left(\theta_{2,3}\right) & 0 \\ 0 & \sin\left(\theta_{2,3}\right) & \cos\left(\theta_{2,3}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\theta_{2,4}\right) & 0 & -\sin\left(\theta_{2,4}\right) \\ 0 & 0 & 1 & 0 \\ 0 & \sin\left(\theta_{2,4}\right) & 0 & \cos\left(\theta_{2,4}\right) \end{pmatrix},
\end{aligned}
\tag{40}
$$

where $c_{i,j} = \cos(\theta_{i,j})$, $s_{i,j} = \sin(\theta_{i,j})$, and

$$P = \begin{pmatrix} p_{y,11} & p_{y,12} & 0 & 0 \\ p_{y,21} & p_{y,22} & 0 & 0 \\ 0 & 0 & p_{x,33} & p_{x,34} \\ 0 & 0 & p_{x,43} & p_{x,44} \end{pmatrix}, \quad q = Q_N Q_X = \begin{pmatrix} q_{11} & q_{12} & 0 & 0 \\ q_{21} & q_{22} & 0 & 0 \\ 0 & 0 & q_{33} & q_{34} \\ 0 & 0 & q_{43} & q_{44} \end{pmatrix},$$

$$Q_{NX}\left(\theta^R\right) = \begin{pmatrix} c_{1,4} & 0 & 0 & -s_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s_{1,4} & 0 & 0 & c_{1,4} \end{pmatrix} \begin{pmatrix} c_{1,3} & 0 & -s_{1,3} & 0 \\ 0 & 1 & 0 & 0 \\ s_{1,3} & 0 & c_{1,3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2,3} & -s_{2,3} & 0 \\ 0 & s_{2,3} & c_{2,3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{2,4} & 0 & -s_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & s_{2,4} & 0 & c_{2,4} \end{pmatrix}. \tag{41}$$

Thus, we need to find $\theta^*_{3,1}, \theta^*_{3,2}, \theta^*_{4,1}$ and $\theta^*_{4,2}$ that make $(3,1), (3,2), (4,1)$ and $(4,2)$ elements of Q_{POGC} to be 0. From (40) and (41), we obtain the equation that the four elements, $(3,1), (3,2), (4,1)$ and $(4,2)$ of Q_{POGC} must satisfy:

$$\begin{aligned}
&[p_{33}(q_{33}s_{1,3} + q_{34}s_{1,4}c_{1,3})] + [p_{34}(q_{43}s_{1,3} + q_{44}s_{1,4}c_{1,3})] = 0, \\
&[p_{43}(q_{33}s_{1,3} + q_{34}s_{1,4}c_{1,3})] + [p_{44}(q_{43}s_{1,3} + q_{44}s_{1,4}c_{1,3})] = 0, \\
&[p_{33}(q_{33}c_{1,3}s_{2,3}c_{2,4} + q_{34}\{-s_{1,4}s_{1,3}s_{2,3}c_{2,4} + c_{1,4}s_{2,4}\})] \\
&\quad + [p_{34}(q_{43}c_{1,3}s_{2,3}c_{2,4} + q_{44}\{-s_{1,4}s_{1,3}s_{2,3}c_{2,4} + c_{1,4}s_{2,4}\})] = 0, \\
&[p_{43}(q_{33}c_{1,3}s_{2,3}c_{2,4} + q_{34}\{-s_{1,4}s_{1,3}s_{2,3}c_{2,4} + c_{1,4}s_{2,4}\})] \\
&\quad + [p_{44}(q_{43}c_{1,3}s_{2,3}c_{2,4} + q_{44}\{-s_{1,4}s_{1,3}s_{2,3}c_{2,4} + c_{1,4}s_{2,4}\})] = 0.
\end{aligned} \tag{42}$$

There are four equations and four unknowns in (42), so a solution can solve for the rotation angles:

$$\theta^{R*} = \left\{\theta^{R*}_{3,1}, \theta^{R*}_{3,2}, \theta^{R*}_{4,1}, \theta^{R*}_{4,2}\right\}.$$

IV. Ilustrative Application

The volatility of international oil prices, particularly following geopolitical events such as Russia's invasion of Ukraine in 2022, has heightened the attention of monetary policy authorities globally on inflation management and inflation expectations. Notably, the relationship between oil prices and inflation expectations has intensified, particularly during the period from 2014 to 2015 compared to earlier years (Badel and McGillicuddy, 2015). Given Korea's heavy reliance on foreign oil, it is crucial to analyze the implications of international oil price fluctuations for the Korean economy.

This section aims to apply the Properly Ordered Zero Sign restriction (POZS) strategy to examine the impact of international oil market shocks on domestic prices, focusing particularly on the role of inflation expectations in the price pass-through mechanism. While there is a body of research examining the relationship between oil prices and inflation in Korea, studies specifically analyzing the role of inflation expectations in the price pass-through process remain scarce. This paper seeks to fill this gap by investigating the effects of international oil market shocks on domestic inflation and assessing the extent to which inflation expectations contribute to this pass-through, utilizing data on inflation expectations provided by the Bank of Korea.

For the empirical analysis, we initially considered a Vector Error Correction Model. However, it is essential to acknowledge that Korea operates as a small open economy, unlike larger economies such as the United States, China, and the EU. In this context, domestic variables, such as Korean prices, do not exert influence over global oil markets. A typical VECM assumes that once the international oil price is included as an endogenous variable, it responds to domestic shocks, which is implausible in the case of small economies.

This inherent conflict in the VECM framework cannot be resolved by merely imposing traditional restrictions (such as short-run, long-run, or zero restrictions). Thus, this study adopts a VECM with Exogenous Variables (VECMX), treating global variables like the international oil price as exogenous rather than endogenous. In the paper, the estimation of the VECMX model employs local projections, which are known to be robust against specification errors (Jorda, 2005). Chong et al. (2012) utilized local projections for estimating VECM.

A significant body of literature has explored the effects of oil price shocks on both small and large economies (Baumeister and Peersman, 2008; Peersman and Robays, 2009; Lippi and Nobili, 2012; Yilmazkuday, 2021; Atems et al., 2015; Liu et al., 2020; Diegel and Nautz, 2021; Cunado et al., 2015). Most of these researches treat oil prices as endogenous variables. Among the extensive research, Boeck and Zoerner (2023) and Wong (2015) specifically analyze the role of inflation expectations in the price pass-through mechanism. Boeck and Zoerner (2023) employed a structural VAR model incorporating both zero and sign restrictions to explore how natural gas prices affect overall prices through inflation expectations using counterfactual analysis. Their findings indicate that natural gas price shocks significantly influence inflation and inflation expectations, with substantial second-round effects arising through inflation expectations. Wong (2015) reported that U.S. inflation expectations are sensitive to real oil price shocks; however, if inflation expectations become less responsive, their role in the pass-through of real oil prices diminishes considerably. Additionally, Baumeister and Peersman (2008), Peersman and Robays (2009), and Lippi and Nobili (2012) examine various types of oil shocks and their macroeconomic transmission mechanisms.

It is critical to utilize the VECMX framework to effectively analyze the impact of global oil market fluctuations on domestic inflation. In analyzing the process through which oil price shocks impact domestic inflation, we aim to determine the extent to which inflation expectations influence this relationship. It is important to note that the first difference of inflation, being an $I(0)$ variable, contrasts with the consumer price index, which is $I(1)$. Thus, inflation expectations must also be treated as $I(0)$. When incorporating inflation expectations into VECMX model, we must carefully consider the implications of mixing $I(1)$ and $I(0)$ variables.

This analysis will contribute to a deeper understanding of the dynamics between international oil prices, domestic inflation, and inflation expectations in the context of a small open economy. By employing the POZS strategy and a VECMX framework, we can gain insights into the mechanisms of price pass-through and the overall implications for monetary policy in Korea.

1. Data

In this study, we employed seven variables to analyze the interactions between Korean domestic and global variables. Among these, four are domestic variables, which include real GDP, Consumer Price Index, interest rates (91-day CD rate which is a short-term interest rate reflecting the return on a certificate of deposit (CD) with a 91-day maturity and Inflation Expectations. The three exogenous global variables include: (1) World Industrial Production Index (OECD IPI): This serves as a proxy for global industrial activity and is critical for understanding international economic conditions. (2) World Oil Production: This measures the total crude oil produced globally, providing insights into supply-side dynamics in the oil market. (3) World Oil Price: Specifically, we used the refiner acquisition cost of imported crude oil, which serves as a reliable indicator of the free market price of imported crude oil. To derive the real acquisition cost of crude oil, we divided the nominal acquisition cost by the U.S. CPI.

Data on Korean Domestic Variables such as real GDP, CPI, interest rates, and inflation expectations were collected from the ECOS database maintained by the Bank of Korea. The OECD IPI data was obtained from the OECD Economic Outlook (OECD EO) dataset. Oil-related data were sourced from the Energy Information Administration (EIA) and the International Energy Agency (IEA). U.S. CPI is sourced from the Bureau of Labor Statistics (BLS).

The analysis covers the sample period from the first quarter of 2002 to the fourth quarter of 2022, encompassing a total of 84 quarters. All variables are treated as $I(1)$ variables and expressed in logarithmic form, except for inflation expectations, which are considered $I(0)$ variables. All data are seasonally adjusted.

2. Impulse Responses Under Properly Ordered Sign Restrictions

This paper calculates the impulse response function using sign restrictions. The domestic shocks are identified as domestic demand (AD) shocks, domestic supply (AS) shocks, monetary policy (MP) shocks, and inflation expectations (IE) shocks, and the global shocks are identified as crude oil shocks, oil supply shocks, and oil demand shocks due to international demand. The specific sign restrictions guiding the impulse response functions are summarized in Table 1, which outlines the expected relationships between the variables in response to the identified shocks:

The first identification strategy emphasizes that global exogenous variables-including world production, crude oil supply, and oil prices- do not react to shocks affecting domestic variables. This principle is necessary for small open economies, where domestic fluctuations typically do not influence global economic conditions. This condition aligns with the findings of Liu et al. (2011) and Carriere-Swallow and Cespedes (2013), who argue that the small open economy assumption implies that international variables remain unaffected by domestic economic shocks. Thus, in the context of our analysis, we maintain that any shocks originating from domestic factors should not cause significant responses in global exogenous variables, ensuring that our identification strategy accurately captures the dynamics of domestic and international interactions. This approach allows us to focus on how domestic inflation and output are influenced by external shocks, particularly those related to the oil market.

Second, identification strategies imposing restrictions on domestic variables are based on established economic theory and previous empirical research:

(1) When Aggregate Supply (AS) shock occurs, a negative correlation between GDP and CPI is imposed, meaning that GDP rises while prices fall. This relationship lasts for approximately two periods, as per Canova and de Nicole (2003), reflecting that supply shocks typically increase output while lowering prices due to improved production efficiency or cost reductions.

(2) When Aggregate Demand (AD) shock occurs, a positive correlation between GDP and CPI is imposed, where both variables rise simultaneously. Interest rates are expected to increase due to price pressures, reflecting the central bank's response to inflation by raising rates. This aligns with standard macroeconomic theory and is consistent with the work of Mountford (2005).

(3) Monetary Policy (MP) shock leads to an increase in interest rates, with a simultaneous decrease in both GDP and CPI. The central bank tightens monetary policy to combat inflation, reducing output and price levels. These constraints are also consistent with Mountford (2005), reflecting typical responses to interest rate hikes aimed at controlling inflation.

(4) No restriction is imposed on Inflation Expectations (IE) shock to allow for a free response across variables. This flexibility is crucial to examine the role of

inflation expectations in the process of price pass-through, particularly how inflation expectations might mediate the relationship between oil price shocks and domestic inflation.

Third, for the identification of different oil shocks. The identification is based on the framework established by Baumeister and Peersman (2008), and Peersman and Robays (2009), who distinguish between different types of oil market shocks, each having distinct economic effects:

(1) Oil Demand Shocks can arise from two different sources:

a) Global Demand (GD) Shocks are oil demand shocks that occur due to increased global economic activity. As the global economy expands, there is an increased demand for oil, leading to higher oil prices. For instance, a global economic boom or strong industrial growth can increase the oil demand.

b) Oil-Specific Demand (OD) Shocks are unfavorable demand shocks specific to the oil market that are not driven by broader economic activity. These shocks can be caused by speculative behavior in oil markets or concerns about future oil supply disruptions, leading to an increase in oil prices without an accompanying increase in global economic activity.

(2) Oil Supply Shocks (OS) occur due to disruptions in the oil supply, resulting in a decrease in oil production and an increase in oil prices. Oil supply shocks can be triggered by various factors, such as geopolitical conflicts (e.g., military conflicts in oil-producing regions) or decisions by oil-producing countries to change production quotas (e.g., decisions made by OPEC). Such shocks directly reduce the available oil supply, pushing prices higher and potentially leading to global economic repercussions.

By distinguishing between these three types of oil market shocks (global demand, oil-specific demand, and oil supply shocks), the model aims to capture the varied channels through which oil price fluctuations can impact both global and domestic economies. This approach ensures that the different nature and origins of oil price changes are reflected in the impulse responses of domestic variables, particularly in the context of a small open economy like Korea.

Fourth, domestic variables respond to international shocks: Given that Korea is a small open economy, its domestic variables are significantly affected by external shocks originating from global markets. This is particularly true for global production and oil market shocks. The model accounts for these external influences, and the following responses are expected for domestic variables in response to international shocks:

(1) When there is a positive shock to world production (i.e., an increase in global industrial output), the following domestic responses are expected: Domestic GDP increases, as higher global economic activity boosts demand for Korean exports, and Domestic CPI increase as it is driven by rising demand and potentially higher input costs as global activity heats up.

(2) An Oil Supply Shock, which reduces the availability of oil on the global market, has adverse effects on the Korean economy: Domestic GDP decreases, due to higher oil prices, which raise production costs and slow economic activity. Domestic CPI increases, as the shock to oil supply pushes energy costs higher, feeding into overall inflation. Inflation Expectations are likely to increase, reflecting concerns about sustained higher prices for energy and other goods tied to oil.

(3) When an oil-specific demand shock occurs, such as speculative buying or concerns about future supply, the following domestic responses are expected: Domestic GDP is likely to decrease, as higher oil prices lead to increased production costs and reduced consumer spending power. Domestic CPI increases, when it is driven by the pass-through of higher oil prices to overall inflation. Inflation Expectations also increase, as consumers and businesses adjust their future price expectations based on the surge in oil prices. The sign restrictions are presented in Table 1.

Table 1. Sign restrictions used for identification

	Shocks						
Response	AS	AD	MP	IE	GD	OS	OD
GDP	+	+	-	*	-	*	*
CPI	-	+	-	+	*	+	+
CD91	*	+	+	*	*	*	*
Inf. Expectations	-	+	-	+	*	*	*
World IPI	0	0	0	0	+	-	-
Oil Production	0	0	0	0	+	-	+
Oil Price	0	0	0	0	+	+	+

Note: A '+' (or '-') sign indicates that the impulse response of the variable in question is restricted to be positive (negative) for two quarters after the shock. A entry with '*' ('0') indicates that no restrictions (zero restrictions) are imposed on the response.

To make the analysis tractable and economically meaningful, we impose sign restrictions that hold for a period of two quarters following each shock. This choice is consistent with prior literature, such as Canova and de Nicole (2003), and reflects the view that the immediate impacts of shocks can be most reliably identified within this short horizon. Although the restrictions are only imposed for two quarters, the impulse responses are computed for a longer horizon of sixteen quarters to capture the medium-term dynamics of the economy. Notably, even after the restrictions are lifted, the direction of the responses often remains stable for a considerable period, suggesting robust effects of the shocks.

Figure A3 presents the impulse responses to various shocks, focusing on the domestic endogenous variables. The core focus of this analysis is on the role of inflation expectations in the pass-through mechanism when oil market shocks occur. Therefore, we will not discuss each set in detail.

The study aims to understand how inflation expectations influence the transmission of oil market shocks (e.g., global oil supply and demand shocks) to domestic inflation. Given that oil prices are a critical input cost for many industries, fluctuations in oil markets can have a substantial impact on overall price levels. Inflation expectations are crucial in this process because they shape both wage-setting behavior and price-setting by firms, thereby amplifying or mitigating the inflationary pressures stemming from oil price changes.

By isolating the role of inflation expectations in this context, we can better understand how inflation expectations contribute to the persistence and magnitude of responses of CPI to oil price shocks.

Figure A4 shows the impulse responses to shocks of exogenous global variables. The effects of the shocks remain robust. That is, even after the restrictions are lifted, the direction of the responses often remains stable for a considerable period.

3. Counterfactual Analysis

In this section, we analyze the influence of inflation expectations on the consumer price index (CPI) response when oil supply shocks and oil-specific demand shocks occur using the Kilian-Lewis counterfactual analysis. This approach allows us to simulate how the CPI would react if the response of expected inflation were controlled or adjusted during oil market shocks.

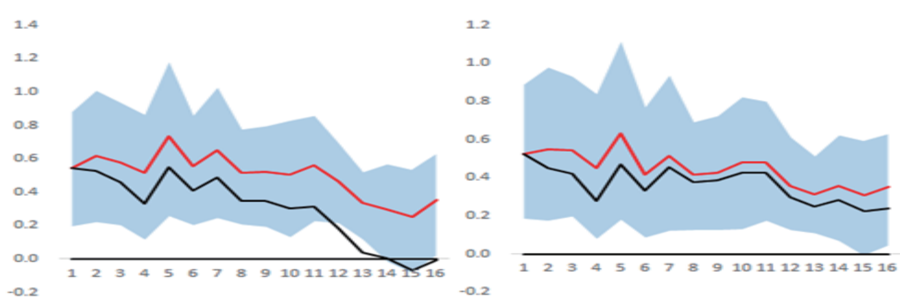
When an oil supply shock occurs, the analysis shows that controlling for the response of inflation expectations leads to a significant decrease in the logarithmized CPI compared to the scenario where expected inflation is left unconstrained. Specifically: After 4 quarters, the CPI decreases by -0.39 %p, after 8 quarters, the CPI decreases by -1.06 %p, after 12 quarters, the CPI decreases by -1.96 %p.

For oil demand shocks, a similar pattern is observed, though the effect is somewhat smaller than in the case of supply shocks: After 4 quarters, the CPI decreases by -0.40 %p, after 8 quarters, the CPI decreases by -0.74 %p, after 12 quarters, the CPI decreases by -0.94 %p (see Table 2, and Figure 1 for a detailed visualization).

Table 2. Cumulative Discrepancies between Original CPI and Counterfactual CPI

	Structural	Shocks
Horizon	Oil Supply	Oil Specific Demand
4	-0.39	-0.40
8	-1.06	-0.74
12	-1.96	-0.94

Figure 1. Counterfactual Analysis: CPI and Counterfactual CPI



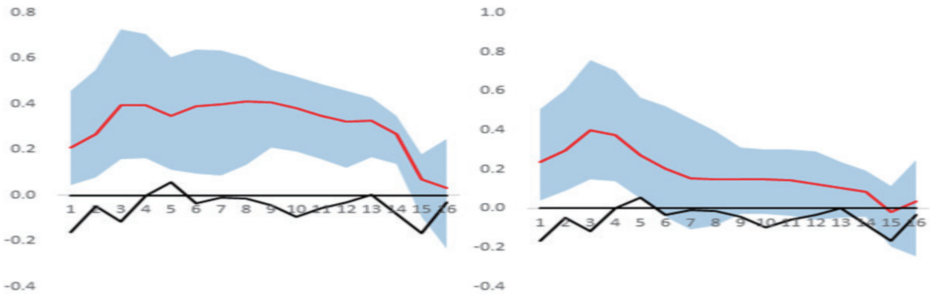
Notes: 1st column for oil supply shock, 2nd column for oil specific demand shock. Shaded Area represents 90 % credible sets. Red real line and black line represent median value and counterfactual values.

These results suggest that when an oil market shock (whether supply or demand) causes oil prices to rise, the CPI is significantly affected by the accompanying increase in inflation expectations. Specifically, expected inflation acts as a price stimulation factor, causing the CPI to rise more than it would if inflation expectations did not increase.

The price stimulation effect of expected inflation is particularly pronounced in the case of oil supply shocks. This may be because supply shocks adjust inflation expectations to a greater extent than demand shocks, and the persistence of their impact is stronger over time.

The analysis demonstrates that inflation expectations play a crucial role in amplifying the pass-through of oil market shocks to domestic inflation. The counterfactual simulations indicate that controlling for expected inflation can significantly reduce the inflationary impact of oil supply and demand shocks, with the effect being more persistent and larger in the case of oil supply shocks (see Figure 2 for a detailed visualization).

Figure 2. Counterfactual Analysis: Inflation Expectations and Adjusted Inflation Expectations



Notes: 1st column for oil supply shock, 2nd column for oil specific demand shock. Shaded Area represents 90 % credible sets. Red real line and black line represent median value and counterfactual values.

V. Summary and Policy Implications

Givens rotation matrices are used for the identification of SVAR models, as different rotation angles yield different Givens matrices, allowing for various identification schemes. Importantly, because the multiplication of Givens matrices is non-commutative (i.e., the order of multiplication matters), the order of multiplication introduces a second degrees of freedom for the identifications. This non-abelianness of matrix multiplication becomes critical when using sign restrictions, particularly for the identification of VARX.

We introduced the properly ordered Givens (POG) matrices, which are the result of multiplying individual Givens matrices in a specific sequence. Proper ordering of multiplication, combined with zero restrictions, allows us to correctly identify the structural shocks in VARX models that include exogenous variables. This method is particularly useful when trying to isolate the effects of exogenous variables (e.g., oil prices, world production) from domestic variables of small open economies.

The illustrative empirical study focused on the impact of oil market shocks on domestic inflation, particularly through the lens of inflation expectations. Using Kilian-Lewis counterfactual analysis, we calculated how the CPI would respond under scenarios where the response of inflation expectations was controlled. The results indicated that inflation expectations significantly amplify the pass-through effect of oil market shocks, particularly oil supply shocks, on domestic prices. This suggests that the inflation expectations channel plays a crucial role in determining the extent to which global oil price shocks affect domestic inflation.

When responses of inflation expectations were controlled, oil supply shocks reduced responses of CPI inflation by -0.39 %p, -1.06 %p, and -1.96 %p after 4, 8, and 12 quarters, respectively. For oil demand shocks, the reduction in CPI inflation was -0.40 %p, -0.74 %p, and -0.94 %p, respectively.

There is a consensus among central banks and academics that anchoring of long-run inflation expectations is important for monetary policy. This is because the measures for long-run inflation expectations provide a key signal of central bank policy credibility. Consequently, the effects of inflation expectations that amplify the responses of inflation to oil price shocks may be mitigated by making the inflation expectations remain near the targets of central banks.

Therefore, our findings highlight the need for monetary authorities to closely monitor and manage inflation expectations. Given that inflation expectations significantly influence the extent of price increases following oil market shocks, stabilizing inflation expectations can help mitigate the overall inflationary impact. In particular, Korea is highly sensitive to external shocks like oil shocks, making it even more important for policymakers to control inflation expectations to mitigate domestic price volatility.

In conclusion, this study underscores the importance of inflation expectations in amplifying the effects of oil price shocks on domestic inflation. It also introduces a novel methodology for identifying structural shocks in SVARX models, which is necessary for analyzing economies that rely heavily on external factors like oil.

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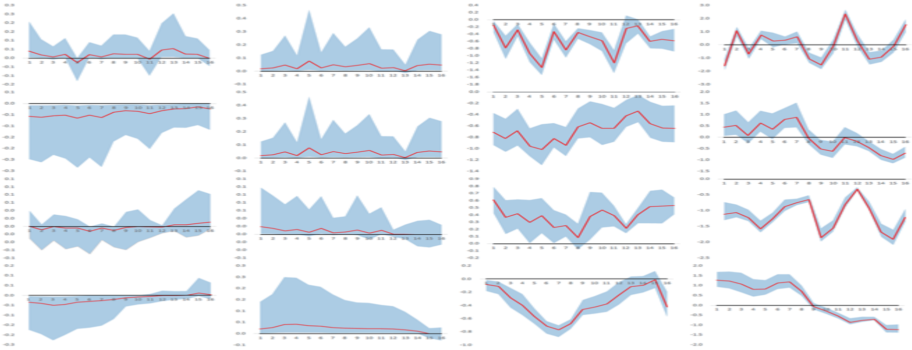
Appendix

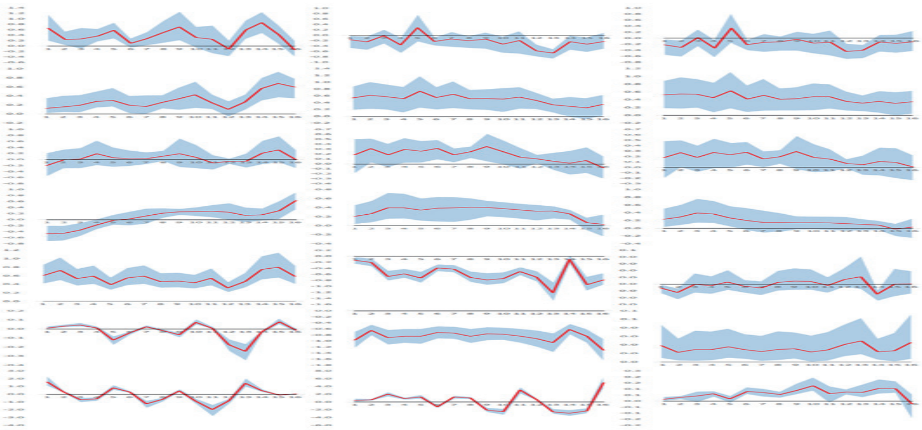
A: Figures

Figure A3. Impulse Response To Shocks of Endogenous Variables



Notes: 1st column for AS shock, 2nd column for AD shock, 3rd column for MP shock, 4th column for I.E. shock; 1st row for response of GDP, 2nd row for response of CPI, 3rd row for response of CD91, 4th row for response of I.E.; Shaded Areas represent 90 % credible sets. Red line is median.

Figure A4. Impulse Response To Shocks of Exogenous Variables



Notes: 1st column for global demand shock, 2nd column for oil supply shock, 3rd column for oil specific demand demand shock; 1st row for response of GDP, 2nd row for response of CPI, 3rd row for response of CD91, 4th row for response of I.E., 5th row for response of global demand, 6th row for response of oil production, 7th row for response of oil price; Shaded Areas represent 90 % credible sets. Red line is median.

B: Counterfactual Analysis (Kilian-Lewis Approach)

We employ Kilian and Lewis (2011) for the counterfactual analysis: Consider the following structural *VAR* model:

$$X_t = \Lambda X_{t-1} + v_t, \tag{43}$$

where

$$X_t = \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \end{bmatrix}, \Lambda = \begin{bmatrix} B_1 & \cdots & B_p \\ I & \cdots & 0 \end{bmatrix}, v_t = \begin{bmatrix} B_0 \varepsilon_t \\ 0 \\ 0 \end{bmatrix}. \tag{44}$$

Let e_j be a $(1 \times N)$ row vector where the value of the jth column is 1 and the values of all remaining elements are 0. When e_j is premultiplied by a matrix B_0, it selectively extracts only the jth row of B_0. In general, the following sandwich form extracts the (i,j) element $b^0_{i,j}$ of B_0.

$$e_i B_0 e_j^{'} = b^0_{ij}. \tag{45}$$

Let $\Psi^k_{j,i}$ be the response function at horizon k of variable j to a 10 percent shock of variable i, then

$$\Psi^k_{j,i} = \frac{e_j (\Lambda)^k (0.1 \times B_0) e_i^{'}}{e_i B_0 e_i^{'}}. \tag{46}$$

The value of jth structural shock ε^j_k that cancels out the response of variable j at horizon k to a 10 % shock of variable i is the solution that satisfies

$$\hat{\Psi}^k_{j,i} = 0 \tag{47}$$

for all $k \in \mathbb{N}$. The response of variable j to a unit shock of variable j is $e_j B_0 e_j^{'}$, thus to offset the response of $e_j\,(0.1\times B_0)\,e_i^{'}/e_iB_0e_i^{'}$, $\hat{\varepsilon}_k^j$ at $k=0$ must satisfy

$$\frac{e_j\,(0.1\times B_0)\,e_i^{'}}{e_iB_0e_i^{'}}+e_jB_0e_j^{'}\hat{\varepsilon}_0^j=0. \tag{48}$$

Since $e_jBe_j^{'}$ is a scalar,

$$\hat{\varepsilon}_0^j=-\frac{1}{e_jBe_j^{'}}\frac{e_j\,(0.1\times B_0)\,e_i^{'}}{e_iB_0e_i^{'}}. \tag{49}$$

The solutions for $k\geq 1$ can be obtained similarly. For $k\in\mathbb{N}$,

$$\Psi_{j,i}^k+\sum_{\ell=0}^{k}e_j\Lambda^{\ell}B_0e_j^{'}\hat{\varepsilon}_{\ell}^j=0. \tag{50}$$

Therefore, we have the solution:

$$\hat{\varepsilon}_k^j=-\frac{\left(\Psi_{j,i}^k+\sum_{\ell=0}^{k-1}e_j\Lambda^{\ell}B_0e_j^{'}\hat{\varepsilon}_{\ell}^j\right)}{e_j\Lambda^kB_0e_j^{'}} \tag{51}$$

for $k\geq 1$.

In the process of calculating the values of $\hat{\varepsilon}_k^j$, the values of other variables are changed. The counterfactual impulse response function $\hat{\Psi}_{j,i}^k$ for i shock is

$$\hat{\Psi}_{j,i}^k=\Psi_{j,i}^k+\sum_{\ell=0}^{k-1}e_j\Lambda^{\ell}B_0e_j^{'}\hat{\varepsilon}_{\ell}^j \tag{52}$$

for $j=1,\cdots,N$.

⟨Abstract in Korean⟩

0 제약된 정배열 기븐스행렬을 이용한 부호제약과 소규모개방경제 VARX에 대한 응용

김기호*

VARX 모형에 기븐스 행렬(Givens matrix)을 이용하여 부호제약을 가하면 내생변수 충격에 대한 반응과 외생변수 충격에 대한 반응이 혼합된다. 내외생 변수 충격에 대한 반응의 혼합현상은 VARX 모형에 대한 기존의 부호제약 방법을 사용할 때 오식별로 귀결된다. 본고는 기븐스행렬이 비가환성을 지니고 있음을 이용해 기븐스행렬을 곱하는 순서를 적절하게 선택하는 한편 0의 제약을 동시에 가함으로써 충격반응의 혼합현상을 해소할 수 있음을 보였다.

본고는 응용 사례로서 국제석유시장의 수요, 공급 충격이 국내물가에 전가되는 과정에서 기대 인플레이션이 어느 정도 영향을 미치는지를 분석해 보았다. 이를 위해 제안된 식별제약조건을 외생변수인 석유시장 관련 변수에 적용하여 석유시장의 공급 및 수요충격을 식별하고, 반사실적 분석을 이용하여 두 외생적 충격요인이 국내 소비자 물가의 반응에 미치는 정도를 분석해 보았다. 분석결과, 석유시장의 수요 및 공급 충격에 대한 소비자물가의 반응은 기대 인플레이션 경로가 작동하는 경우 그렇지 않은 경우에 비해 더 확대되는 것으로 나타났다.

기대 인플레이션은 중앙은행 정책의 신뢰도에 대한 중요한 시그널 역할을 하는 것으로 알려져 있다. 기대 인플레이션 경로를 통해 발생하는 인플레이션 확대효과를 완화하기 위해서는 통화정책의 신뢰도와 직결된 기대 인플레이션에 대한 모니터링을 적극적으로 실시하고 기대의 안정화 등에 더욱 노력할 필요가 있다.

핵심 주제어: 외생성, 부호제약, 0의 제약

JEL Classification: C13, C21, C23

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논고 작성에 많은 도움을 주신 이재원 경제연구원 원장, 조태형 부원장, 박성호 강원기획조사부장, 소인환 국제경제연구실장, 김도완 경남총무팀장, 한상범 경기대학교 경제학부 교수 및 원내 세미나 참석자분들께 감사드린다. 남아있는 오류는 저자의 책임임을 밝힌다.

BOK 경제연구 발간목록

한국은행 경제연구원에서는 Working Paper인 『BOK 경제연구』를 수시로 발간하고 있습니다. 『BOK 경제연구』는 주요 경제 현상 및 정책 효과에 대한 직관적 설명 뿐 아니라 깊이 있는 이론 또는 실증 분석을 제공함으로써 엄밀한 논증에 초점을 두는 학술논문 형태의 연구이며 한국은행 직원 및 한국은행 연구용역사업의 연구 결과물이 수록되고 있습니다.
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22	Identifying Government Spending Shocks and Multipliers in Korea	Kwangyong Park · Eun Kyung Lee
23	Systemic Risk of the Consumer Credit Network across Financial Institutions	Hyun Hak Kim · Hosung Jung
24	Impact of Chinese Renminbi on Korean Exports: Does Quality Matter?	Jihyun Eum
25	Uncertainty, Credit and Investment: Evidence from Firm-Bank Matched Data	Youngju Kim · Seohyun Lee · Hyunjoon Lim
26	A Structural Change in the Trend and Cycle in Korea	Nam Gang Lee · Byoung Hoon Seok

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2	달러라이제이션이 확산된 북한경제에서 보유외화 감소가 물가·환율에 미치는 영향	문성민 · 김병기
3	상태공간 벡터오차수정모형을 이용한 월별 GDP 추정: 깁스표본추출 접근	김기호
4	우리나라 외환시장 오퍼레이션의 행태 및 환율변동성 완화 효과	박준서 · 최경욱
5	Common Factor Augmented Forecasting Models for the US Dollar-Korean Won Exchange Rate	Hyeongwoo Kim · Soohyon Kim
6	북한「경제연구」로 분석한 경제정책 변화: 텍스트 마이닝 접근법	김수현 · 손 욱
7	북한의 광물 수출과 품목별 수입: 대중무역을 중심으로	김병연 · 김민정 · 김다울
8	Network-Based Measures of Systemic Risk in Korea	Jaewon Choi · Jieun Lee
9	Aggregate Productivity Growth and Firm Dynamics in Korean Manufacturing 2007-2017	Kyoo il Kim · Jin Ho Park
10	2001년 이후 한국의 노동생산성 성장과 인적자본: 교육의 질적 개선 효과를 중심으로	유혜미
11	House Prices and Household Consumption in Korea	Seungyoon Lee
12	글로벌 가치사슬 변화가 경제성장에 미치는 영향: 2008년 금융위기 전후 전·후방참여 효과의 국제비교를 중심으로	김세완 · 최문정
13	산업구조조정이 고용 및 성장에 미치는 영향	서병선 · 김태경
14	Cross-border Trade Credit and Trade Flows During the Global Financial Crisis	Moon Jung Choi · Sangyeon Hwang · Hyejoon Im

15	International Co-movements and Determinants of Public Debt	Hasan Isomitdinov · Vladimir Arčabić · Junsoo Lee · Youngjin Yun
16	북한 비공식금융 실태조사 및 분석·평가	이주영 · 문성민
17	북한의 장기 경제성장률 추정: 1956~1989년	조태형 · 김민정
18	Macroeconomic and Financial Market Analyses and Predictions through Deep Learning	Soohyon Kim
19	제조업의 수출과 생산성 간 관계 분석: 사업체 자료 이용	이윤수 · 김원혁 · 박진호
20	우리나라 제조업 수출기업의 내수전환 결정요인 분석	남윤미 · 최문정
21	A Model of Satisficing Behaviour	Rajiv Sarin · Hyun Chang Yi
22	Vulnerable Growth: A Revisit	Nam Gang Lee
23	Credit Market Frictions and Coessentiality of Money and Credit	Ohik Kwon · Manjong Lee
24	북한의 자본스톡 추정 및 시사점	표학길 · 조태형 · 김민정
25	The Economic Costs of Diplomatic Conflict	Hyejin Kim · Jungmin Lee
26	Central Bank Digital Currency, Tax Evasion, Inflation Tax, and Central Bank Independence	Ohik Kwon · Seungduck Lee · Jaevin Park
27	Consumption Dynamics and a Home Purchase	Dongjae Jung
28	자본유입과 물가상승률 간의 동태적 상관관계 분석: 아시아의 8개국 소규모 개방경제를 중심으로	최영준 · 손종칠
29	The Excess Sensitivity of Long-term Interest rates and Central Bank Credibility	Kwangyong Park

30	Wage and Employment Effects of Immigration: Evidence from Korea	Hyejin Kim
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2	한국경제의 추세 성장률 하락과 원인	석병훈・이남강
3	Financial Globalization: Effects on Banks' Information Acquisition and Credit Risk	Christopher Paik
4	The Effects of Monetary Policy on Consumption: Workers vs. Retirees	Myunghyun Kim・Sang-yoon Song
5	북한지역 토지자산 추정에 관한 연구: 프레임워크 개발 및 탐색적 적용	임송
6	김정은 시대 북한의 금융제도 변화 – 북한 문헌 분석을 중심으로 –	김민정・문성민
7	Chaebols and Firm Dynamics in Korea	Philippe Aghion・Sergei Guriev・Kangchul Jo
8	한국의 화폐환상에 관한 연구	권오익・김규식・황인도
9	재원조달 방법을 고려한 재정지출 효과 분석 : 미국의 사례를 중심으로	김소영・김용건
10	The Impact of Geopolitical Risk on Stock Returns: Evidence from Inter-Korea Geopolitics	Seungho Jung・Jongmin Lee・Seohyun Lee
11	Real Business Cycles in Emerging Countries: Are Asian Business Cycles Different from Latin American Business Cycles?	Seolwoong Hwang・Soyoung Kim
12	우리 수출의 글로벌 소득탄력성 하락 요인 분석	김경근
13	북한의 경제체제에 관한 연구: 실태와 평가	양문수・임송
14	Distribution-Dependent Value of Money: A Coalition-Proof Approach to Monetary Equilibrium	Byoung-Ki Kim・Ohik Kwon・Suk Won Lee

15	A Parametric Estimation of the Policy Stance from the Central Bank Minutes	Dong Jae Jung
16	The Immigrant Wage Gap and Assimilation in Korea	Hyejin Kim · Chulhee Lee
17	Monetary Non-Neutrality in a Multisector Economy: The Role of Risk-Sharing	Jae Won Lee · Seunghyeon Lee
18	International Transmission of Chinese Monetary Policy Shocks to Asian Countries	Yujeong Cho · Soyoung Kim
19	The Impact of Robots on Labor Demand: Evidence from Job Vacancy Data for South Korea	Hyejin Kim
20	전공 불일치가 불황기 대졸 취업자의 임금에 미치는 장기 효과 분석	최영준
21	Upstream Propagation of the U.S.-China Trade War	Minkyu Son
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2	Transmission of Global Financial Shocks: Which Capital Flows Matter?	Bada Han
3	Measuring the Effects of LTV and DTI Limits: A Heterogeneous Panel VAR Approach with Sign Restrictions	Soyoung Kim · Seri Shim
4	A Counterfactual Method for Demographic Changes in Overlapping Generations Models	Byongju Lee
5	Housing Wealth, Labor Supply, and Retirement Behavior: Evidence from Korea	Jongwoo Chung
6	Demand Shocks vs. Supply Shocks: Which Shocks Matter More in Income and Price Inequality?	Seolwoong Hwang · Kwangwon Lee · Geunhyung Yim

7	Financial Literacy and Mutual Fund Retail Investing: Evidence from Korea During the 2008 Financial Crisis	Jongwoo Chung · Booyuel Kim
8	Exchange Rate Regime and Optimal Policy: The Case of China	Yujeong Cho · Yiping Huang · Changhua Yu
9	북한 수출입단가지수 추정: 북중무역 데이터를 중심으로	이종민 · 김민정
10	탄소배출을 감안한 국가별 녹색 총요소생산성 분석	안상기
11	북한 소비자 지급수단 조사 및 분석	이주영
12	Selection into Outsourcing versus Integration Strategies for Heterogeneous Multinationals	Sangho Shin
13	Central Bank Digital Currency and Privacy: A Randomized Survey Experiment	Syngjoo Choi · Bongseop Kim · Young Sik Kim · Ohik Kwon
14	Technological Change, Job Characteristics, and Employment of Elderly Workers: Evidence from Korea	Jongwoo Chung · Chulhee Lee
15	Machine-Learning-Based News Sentiment Index (NSI) of Korea	Beomseok Seo · Younghwan Lee · Hyungbae Cho
16	빅데이터를 이용한 실시간 민간소비 예측	신승준 · 서범석
17	Fixed Effects Quantile Estimations with Extended Within Transformation and their Application	Ki-Ho Kim
18	글로벌 금융위기 이후 가계소비행태 변화 분석: 세대별 소비행태를 중심으로	최영준
19	Optimal Monetary Policy under Heterogeneous Consumption Baskets	Seunghyeon Lee

20	통화정책 충격이 생산과 물가에 미치는 효과의 국가별 차이 및 결정요인	임근형 · 나승호 · 오다운
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2	소득동질혼과 가구구조가 가구소득 불평등에 미치는 영향: 국제비교를 중심으로	박용민 · 허 정
3	Dominant Currency Pricing: Evidence from Korean Exports	Minkyu Son
4	Banking Crisis, Venture Capital and Innovation	Chun–Yu Ho · Won Sung
5	Can Robots Save Workers? The Effects of Robots on Workplace Injuries and Workers' Health in Korea	Hyejin Kim
6	International Reserve Accumulation: Balancing Private Inflows with Public Outflows	Bada Han · Dongwook Kim · Youngjin Yun
7	Global Bank Branches and Financial Stability: How Do Global Bank Branches Amplify Financial Shocks?	Yoocheol Noh
8	인구구조 변화에 따른 산업별 고용인력 변화와 정책대안별 효과 추정: 여성, 고령자, 외국인 고용확대를 중심으로	김혜진 · 정종우
9	북한 장기 수출입 데이터 재구축 및 분석 : 1962~2018년	김민정 · 김다울
10	Econometric Forecasting Using Ubiquitous News Text: Text–enhanced Factor Model	Beomseok Seo
11	Changes in Inflation Dynamics in Korea: Global Factor, Country Factor, and their Propagation	Yun Jung Kim · Noh–Sun Kwark
12	Financial Technologies and the Effectiveness of Monetary Policy Transmission	Iftekhar Hasan · Boreum Kwak·Xiang Li

13	북한의 시장물가: 2006~2022	임 송 · 문승현
14	지난 60년 경제환경변화와 한국기업 재무지표 변화: 『기업경영분석』(1961-2021)에 나타난 지표를 중심으로 Korea's Economic Policy Changes: Reflected in the Corporate Financial Indicators During the Last 60 Years	조윤제 · 최연교
15	Extended Two-Way Fixed Effects Quantile Cointegration Regression and Its Application	Ki-Ho Kim
16	In Search of the Origin of Original Sin Dissipation	Bada Han · Jangyoun Lee · Taehee Oh
17	대규모·비선형 베이지안 VAR 모형을 활용한 한국 거시경제 전망 및 시나리오 분석	강규호 · 김도완
18	Does the Uncovered Interest Parity Hold in Korea?	Joonyoung Hur · Kwanho Shin
19	북한이탈주민의 건강과 경제적 적응에 대한 연구: 국민건강정보DB 분석을 중심으로	정승호 · 위혜승 · 이종민
20	The Credit-Driven Business Cycles in South Korea: How Important is the Credit Supply Channel?	Nam Gang Lee · Seungho Nah
21	The Effects of Monetary Policy Shocks on Inflation Heterogeneity: The Case of Korea	Seolwoong Hwang
22	Dollar and Government Bond Liquidity: Evidence from Korea	Jieun Lee
23	우리나라의 가계부채와 소득불평등	김수현 · 황설웅
24	초저출산의 경제적·비경제적 원인: 설문 실험을 통한 분석	남윤미 · 황인도
25	한국경제 80년(1970-2050) 및 미래 성장전략 Eighty Years of the Korean Economy (1970-2050): The Past and a Future Growth Strategy	조태형